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published in
Network-Oriented Modeling for Adaptive Networks: Designing Higher-Order Adaptive Biological, Mental and Social Network Models.
2020

DOI (link to publisher)
10.1007/978-3-030-31445-3_11

document version
Publisher's PDF, also known as Version of record

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Chapter 11
Relating Emerging Network Behaviour to Network Structure

Abstract Emerging behaviour of a network is a consequence of the network's structure. However, it may often not be easy to find out how this relation between structure and behaviour exactly is. In this chapter, results are presented on how certain properties of network structure determine network behaviour. The network structure characteristics considered include both connectivity characteristics in terms of being strongly connected, and aggregation characteristics in terms of properties of combination functions to aggregate multiple impacts on a state. In particular, results are found for networks that are strongly connected and combination functions that are strictly monotonically increasing and scalar-free. This class of combination functions includes linear combination functions such as scaled sum functions but also nonlinear ones such as Euclidean combination functions of any order *n* and scaled geometric mean combination functions. In addition, some results are found on how timing characteristics affect final outcomes of the network behaviour.

Keywords Network structure · Social contagion · Asymptotic network behavior · Social convergence · Mathematical analysis

11.1 Introduction

The emerging behaviour of networks is often considered an interesting and sometimes fascinating consequence of the network's structure. Although the emerging behaviour is entailed by the network structure, finding the relation between network structure and network behaviour may be a real challenge. Often simulations under varying settings of the structure characteristics are used to reveal just a glimpse of this relation. But sometimes it is possible to find out how certain properties of the emerging behaviour can be derived in a mathematical manner from certain characteristics of the network structure. In this chapter such cases are shown, using the Network-Oriented Modeling approach from Chap. 2 and (Treur 2016b, 2019) as a vehicle. This approach enables to derive theoretical results that predict emerging behavior that is observed in specific cases of simulations.

J. Treur, *Network-Oriented Modeling for Adaptive Networks: Designing Higher-Order Adaptive Biological, Mental and Social Network Models*, Studies in Systems, Decision and Control 251, https://doi.org/10.1007/978-3-030-31445-3_11

Network structure is in principle described by a number of characteristics that, as considered here, concern: (1) *connectivity* characteristics, describing how different parts of the network connect, (2) *aggregation* characteristics, describing how multiple connections to the same node are handled, and (3) *timing* characteristics, describing how fast network states change over time. For temporal-causal networks, more specifically, such characteristics relate to connection weights defining the connectivity, combination functions defining aggregation of the impacts of multiple states on a given state, and speed factors defining the speed of change of a state. The challenge then is to find out how properties of connection weights, combination functions and speed factors relate to emerging behavior.

In particular, in this chapter it will be addressed what behaviour emerges concerning equilibrium states that are reached; for example:

- What are bounds between which in the end equilibrium values of states occur?
- How much variation occurs for these equilibrium values?
- Under which conditions a common equilibrium value for the different states occurs?

Answers for such questions can be relevant, for example, to predict the spread of information or opinions, or social contagion of emotions; e.g. (Bosse et al. 2015; Castellano et al. 2009).

As a result of the mathematical analysis performed, a number of properties of a network structure have been identified such that any network with a structure satisfying these properties show similar emerging behavior. These structure properties include connectivity characteristics of the network and aggregation characteristics in terms of properties of combination functions used to aggregate the impact of multiple incoming connections to a node. The identified properties of the combination functions define a class of functions most of which are nonlinear, although linear functions are still included. Among the nonlinear ones are Euclidean combination functions of any order n, and scaled geometric mean combination functions. Examples of combination functions that do not belong to this class are minimum and maximum combination functions and logistic sum combination functions. Also some results are presented on how timing characteristics of the network affect the outcomes of the network's behaviour.

In this chapter, in Sect. 11.2 basic concepts are introduced. Section 11.3 shows simulation examples of the emerging behaviour phenomena that can be observed. In Sect. 11.4 properties of network structure are defined that are relevant for the considered types of emerging behaviour. Section 11.5 discusses a number of results for the relation between network structure and network behaviour addressing the questions above. These have been proven mathematically; proofs are included in Chap. 15, Sect. 15.6. Section 11.6 examines a further set of simulations, this time with focus on Euclidean, scaled geometric mean and scaled maximum combination functions. A result is found relating Euclidean combination functions of very high order n to scaled maximum combination functions. Section 11.7 is a final discussion.

11.2 Conceptual and Numerical Representation of a Network

The modeling perspective (Treur 2016b, 2019) used in this chapter, interprets connections in a network in terms of causality and dynamics; see Chap. 2. In the *temporal-causal networks* used, nodes in a network are interpreted as states that vary over time, and the connections are interpreted as causal relations that define how each state can affect other states over time. To define such a network structure, three main elements have to be addressed:

(a) connectivity (the connections in the network)
(b) aggregation (how multiple connections to one node are aggregated)
(c) timing (how timing of the different states takes place).

These three notions determine the characteristics of the network structure. For temporal-causal networks they are modeled by connection weights, combination functions, and speed factors, respectively, which is summarized as:

(a) **Connectivity**

- connection weights from a state X to a state Y, denoted by $\boldsymbol{\omega}_{X,Y}$

(b) **Aggregation**

- a combination function for each state Y, denoted by $\mathbf{c}_Y(..)$

(c) **Timing**

- a speed factor for each state Y, denoted by $\boldsymbol{\eta}_Y$.

Based on these, a conceptual representation of a temporal-causal network model includes labels for connection weights, combination functions, and speed factors; see the upper part (first 5 rows) of Table 11.1. Note that in the current chapter only networks with nonnegative connection weights are considered.

Combination functions are similar to the functions used in a static manner in the (deterministic) Structural Causal Model perspective described, for example, in (Pearl 2000). However, here they are used in a dynamic manner. For example, (Pearl 2000), p. 203, denotes nodes by V_i and the functions corresponding to combination functions by f_i. Pearl (2000) also points at the problem of underspecification for aggregation of multiple connections, as in the often used graph representations the role of combination functions f_i for nodes V_i, is lacking, and they are therefore not a full specification of the network structure.

To provide sufficient flexibility, for each state a specific combination function can be chosen to specify how multiple causal impacts on this state are aggregated. A number of standard combination functions are available as options in the *combination function library* currently including up to tens of functions, but also new functions can be added to the library.

Table 11.1 Conceptual and numerical representations of a temporal-causal network model)

Concept	Conceptual representation	Explanation
States and connections	X, Y, $X \rightarrow Y$	Describes the nodes and links of a network structure (e.g., in graphical or matrix format)
Connection weight	$\boldsymbol{\omega}_{X,Y}$	The *connection weight* $\boldsymbol{\omega}_{X,Y} \in [-1, 1]$ represents the strength of the causal impact of state X on state Y through connection $X \rightarrow Y$
Aggregating multiple impacts on a state	$\mathbf{c}_Y(..)$	For each state Y (a reference to) a *combination function* $\mathbf{c}_Y(..)$ is chosen to combine the causal impacts of other states on state Y
Timing of the effect of causal impact	$\boldsymbol{\eta}_Y$	For each state Y a *speed factor* $\boldsymbol{\eta}_Y \geq 0$ is used to represent how fast a state is changing upon causal impact
Concept	**Numerical representation**	**Explanation**
State values over time t	$Y(t)$	At each time point t each state Y in the model has a real number value, usually in [0, 1]
Single causal impact	$\mathbf{Impact}_{X,Y}(t) = \boldsymbol{\omega}_{X,Y}\, X(t)$	At t state X with a connection to Y has an impact on Y, using connection weight $\boldsymbol{\omega}_{X,Y}$
Aggregating multiple causal impacts	$\mathbf{aggimpact}_Y(t)$ $= \mathbf{c}_Y\left(\mathbf{impact}_{X_1,Y}(t), \ldots, \mathbf{impact}_{X_k,Y}(t)\right)$ $= \mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\right)$	The aggregated causal impact of $k \geq 1$ states $X_1, \ldots, X_k$ on Y at t, is determined using combination function $\mathbf{c}_Y(..)$
Timing of the causal effect	$Y(t+\Delta t) = Y(t) +$ $\boldsymbol{\eta}_Y[\mathbf{aggimpact}_Y(t) - Y(t)]\Delta t$ $= Y(t) +$ $\boldsymbol{\eta}_Y\left[\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\right) - Y(t)\right]\Delta t$	The causal impact on Y is exerted over time gradually, using speed factor $\boldsymbol{\eta}_Y$; here the X_i are all $k \geq 1$ states with outgoing connections to state Y

The lower part of Table 11.1 shows the numerical representation describing the dynamics of a temporal-causal network by difference equations, defined on the basis of the network structure as described by the upper part of that table. Thus dynamic semantics is associated in a numerical-mathematically defined manner to any conceptual temporal-causal network specification. This provides a well-defined relation between network structure and network dynamics at the base level. The difference equations in Table 11.1 last row can be used both for simulation and for mathematical analysis. In Fig. 11.1 the basic relation between structure and dynamics is indicated by the horizontal arrow in the lower part representing the base level; see also in Chap. 2, Fig. 2.6. The upper part will be addressed in Sects. 11.4, 11.5 and 11.6.

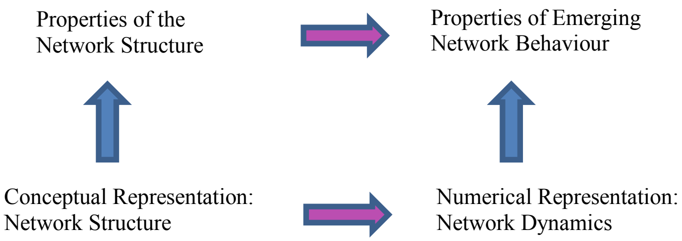

Fig. 11.1 Bottom layer: the conceptual representation defines the numerical representation. Top layer: properties of network structure entail properties of emerging network behaviour

Often used examples of combination functions are the *identity* function **id(.)** for states with impact from only one other state, the scaled maximum and minimum function $\mathbf{smax}_\lambda$(..) and $\mathbf{smin}_\lambda$(..), the *scaled sum* function $\mathbf{ssum}_\lambda$(.), and the scaled geometric mean function $\mathbf{sgeomean}_\lambda$(.), all with scaling factor $\boldsymbol{\lambda}$, and the *advanced logistic sum* combination function $\mathbf{alogistic}_{\sigma,\tau}$(..) with steepness $\boldsymbol{\sigma}$ and threshold $\boldsymbol{\tau}$:

$$
\begin{aligned}
&\mathbf{id}(V) = V\\
&\mathbf{smax}_\lambda(V_1, \ldots, V_k) = \mathbf{max}(V_1, \ldots, V_k)/\boldsymbol{\lambda}\\
&\mathbf{smin}_\lambda(V_1, \ldots, V_k) = \mathbf{min}(V_1, \ldots, V_k)/\boldsymbol{\lambda}\\
&\mathbf{ssum}_\lambda(V_1, \ldots, V_k) = (V_1 + \cdots + V_k)/\boldsymbol{\lambda}\\
&\mathbf{sgeomean}_\lambda(V_1, \ldots, V_k) = \sqrt[k]{\frac{V_1 * \ldots * V_k}{\boldsymbol{\lambda}}}\\
&\mathbf{alogistic}_{\sigma,\tau}(V_1, \ldots, V_k) = \left[\frac{1}{1+\mathrm{e}^{-\boldsymbol{\sigma}(V_1 + \cdots + V_k - \boldsymbol{\tau})}} - \frac{1}{1+\mathrm{e}^{\boldsymbol{\sigma\tau}}}\right](1+\mathrm{e}^{-\boldsymbol{\sigma\tau}})
\end{aligned} \tag{11.1}
$$

In addition to the above functions, generalising the scaled sum function, a *Euclidean combination function* is defined as

$$\mathbf{eucl}_{n,\lambda}(V_1, \ldots, V_k) = \sqrt[n]{\frac{V_1^n + \cdots + V_k^n}{\boldsymbol{\lambda}}} \tag{11.2}$$

where n is the *order* (which can be any positive natural number but also any positive real number), and $\boldsymbol{\lambda}$ is again a scaling factor. Note that indeed for $n = 1$ (first order) we get the scaled sum function

$$\mathbf{eucl}_{1,\lambda}(V_1, \ldots, V_k) = \mathbf{ssum}_\lambda(V_1, \ldots, V_k) \tag{11.3}$$

For $n = 2$ it is the second-order Euclidean combination function defined by

$$\mathbf{eucl}_{2,\lambda}(V_1, \ldots, V_k) = \sqrt{\frac{V_1^2 + \cdots + V_k^2}{\boldsymbol{\lambda}}} \tag{11.4}$$

This second-order Euclidean combination function often occurs in aggregating the error value in optimisation and in parameter tuning using the root-mean-square deviation (RMSD).

For very high values of the order n the limit of an nth order normalised Euclidean function with scaling factor

$$\boldsymbol{\lambda}(n) = \boldsymbol{\omega}_{X_1,Y}{}^{n} + \cdots + \boldsymbol{\omega}_{X_k,Y}{}^{n} \tag{11.5}$$

is a normalised scaled maximum function with scaling factor $\max\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right)$:

$$\lim_{n\to\infty} \mathbf{eucl}_{n,\boldsymbol{\lambda}(n)}(V_1, \ldots, V_k) = \mathbf{smax}_{\max\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)}(V_1, \ldots, V_k) \tag{11.6}$$

This will be shown both by simulation and by mathematical analysis later in Sect. 11.6.

11.3 Examples of a Network's Emerging Behaviour

In this section a few examples of a Social Network for social contagion are discussed. They all concern a fully connected network.

11.3.1 The Example Social Network

The example network is shown in Fig. 11.2 and has connection weights and speed factors as shown in the role matrices in Box 11.1, with initial values shown in Table 11.2. This is actually the same example as used in Chap. 2 for a first analysis.

Fig. 11.2 The example social network

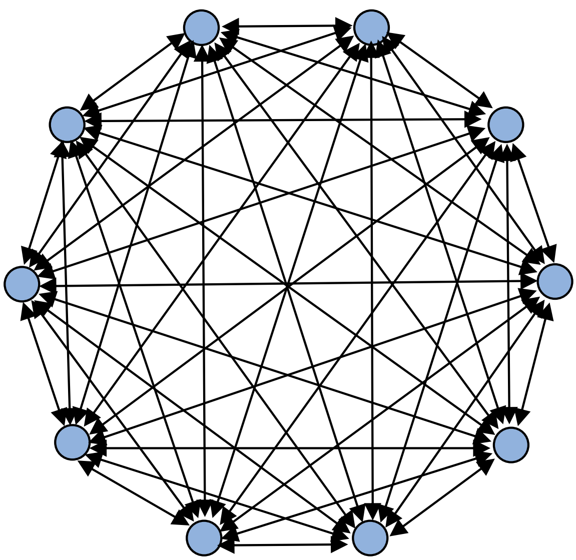

Table 11.2 Initial values for the example

Initial values

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
0.1	0.3	0.9	0.8	0.5	0.6	0.85	0.05	0.25	0.4

Box 11.1 Role matrices for the example network

mb base connectivity	1	2	3	4	5	6	7	8	9
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_1	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_3	X_1	X_2	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_4	X_1	X_2	X_3	X_5	X_6	X_7	X_8	X_9	X_{10}
X_5	X_1	X_2	X_3	X_4	X_6	X_7	X_8	X_9	X_{10}
X_6	X_1	X_2	X_3	X_4	X_5	X_7	X_8	X_9	X_{10}
X_7	X_1	X_2	X_3	X_4	X_5	X_6	X_8	X_9	X_{10}
X_8	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_9	X_{10}
X_9	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_{10}
X_{10}	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9

mcw connection weights	1	2	3	4	5	6	7	8	9
X_1	0.25	0.1	0.25	0.25	0.25	0.2	0.1	0.25	0.2
X_2	0.1	0.25	0.15	0.2	0.1	0.1	0.25	0.15	0.25
X_3	0.2	0.25	0.25	0.1	0.25	0.2	0.1	0.25	0.2
X_4	0.1	0.2	0.1	0.2	0.25	0.15	0.25	0.15	0.2
X_5	0.2	0.1	0.2	0.15	0.25	0.2	0.05	0.2	0.1
X_6	0.15	0.2	0.15	0.8	0.25	0.2	0.15	0.1	0.2
X_7	0.1	0.15	0.1	0.25	0.2	0.1	0.25	0.2	0.15
X_8	0.25	0.25	0.25	0.15	0.1	0.25	0.2	0.15	0.8
X_9	0.25	0.25	0.1	0.25	0.2	0.25	0.15	0.1	0.2
X_{10}	0.1	0.25	0.15	0.25	0.15	0.1	0.25	0.25	0.15

mcfw combination function weights	1 eucl	2 alogistic
X_1	1	
X_2	1	
X_3	1	
X_4	1	
X_5	1	
X_6	1	
X_7	1	
X_8	1	
X_9	1	
X_{10}	1	

mcfp function parameter	1 eucl		2 alogistic	
	1	2	1 σ	2 τ
X_1	1	1.85		
X_2	1	1.55		
X_3	1	1.8		
X_4	1	1.6		
X_5	1	1.45		
X_6	1	2.2		
X_7	1	1.5		
X_8	1	2.4		
X_9	1	1.75		
X_{10}	1	1.65		

ms speed factors	1
X_1	0.8
X_2	0.5
X_3	0.8
X_4	0.5
X_5	0.5
X_6	0.5
X_7	0.8
X_8	0.5
X_9	0.5
X_{10}	0.5

11.3.2 Three Simulations with Different Emerging Behaviour

For this example Social Network simulations have been performed for two types of combination functions: scaled sum and advanced logistic sum combination functions. In Sect. 11.6 similar simulations will be shown for scaled geometric mean, Euclidean, and scaled maximum combination functions. Figure 11.3 shows three different example simulations (all with step size $\Delta t = 0.25$):

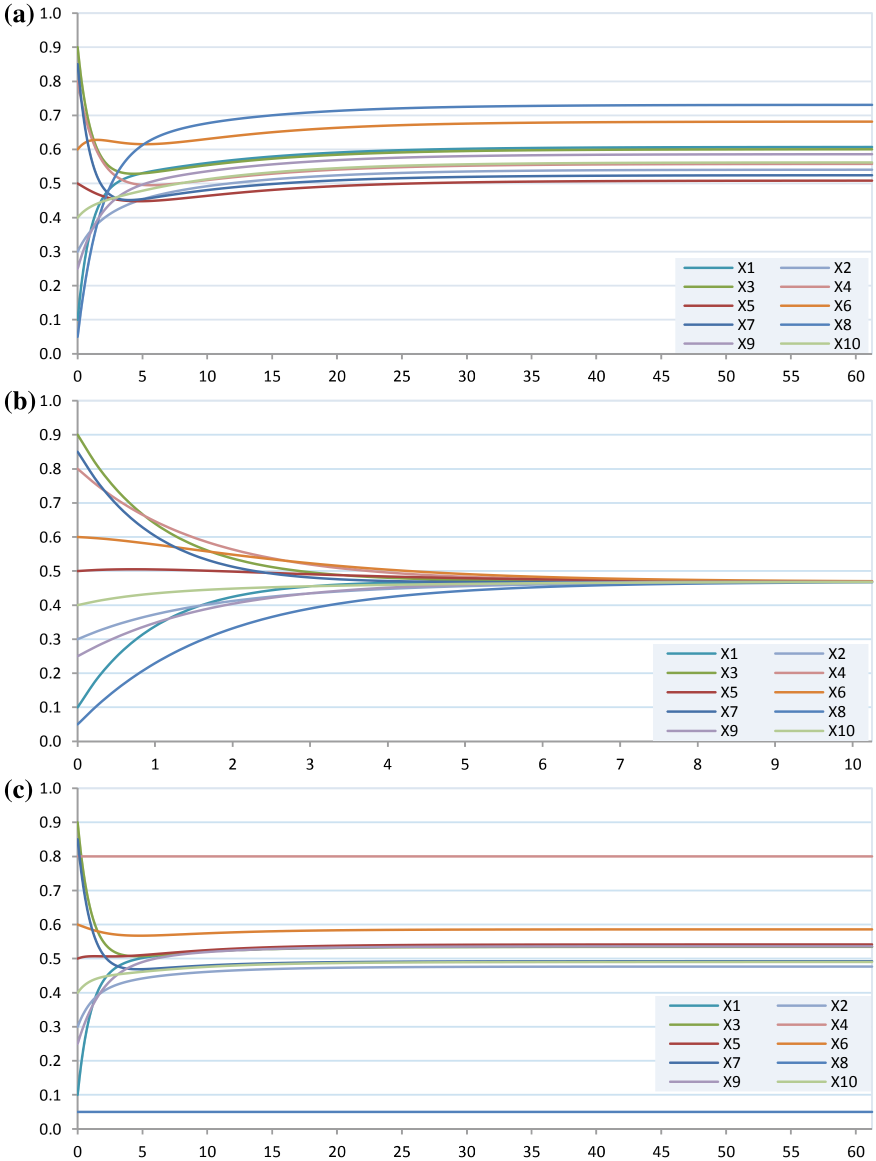

Fig. 11.3 The example network of Fig. 11.2 with **a** upper graph: advanced logistic sum combination functions with steepness **σ** = 1.5, threshold **τ** = 0.3 (no common equilibrium value), **b** middle graph: normalised scaled sum functions (common equilibrium value), **c** lower graph: normalised scaled sum functions with constant X_4 (at 0.8) and X_8 (at 0.05) (no common value)

- in the upper graph advanced logistic sum combination functions are used,
- in the middle graph normalized scaled sum functions, and
- in the lower graph scaled sum functions while two states are independent and remain constant.

It turns out that in one of the three cases convergence to a common equilibrium value takes place, but not in the other two cases; instead some (imperfect) form of clustering seems to take place. How can we explain these differences from the structure of the networks? This question will be answered in Sect. 11.5.

11.4 Relevant Network Structure Characteristics

As explained in Sect. 11.2 and Table 11.1 the basic difference equations describing network dynamics relate to network structure. This covers the lower part of Fig. 11.1. As emerging behaviour shows itself over longer time durations, and the difference equations describe the very small steps in the dynamics, to relate emerging network behaviour to network structure, the gap between these small steps and longer time durations has to be bridged. How that can be done is discussed in the current section and in Sect. 11.5. Here it is discussed which properties of network structure (see Fig. 11.1, left upper corner) underly the behavioural differences (right upper corner in Fig. 11.1) shown in Sect. 11.3 and Fig. 11.3. Proofs can be found in Chap. 15, Sect. 15.6.

Properties of all three main elements of the network's structure (connectivity, aggregation, and timing) have turned out relevant as determining factors for the network's emerging behaviour. Properties of the network's *aggregation* are expressed by combination functions, and properties of the network's *connectivity* are expressed by the connections and their weights. First, in Sect. 11.4.1 the relevant properties of combination functions for aggregation are addressed, and next in Sect. 11.4.2 relevant properties of the network's connectivity. In Sect. 11.5.4 the third main element of a network's structure determining the network's behaviour, namely *timing*, is analysed as well.

11.4.1 Relevant Network Aggregation Characteristics in Terms of Properties of Combination Functions

For the combination functions describing the network's aggregation characteristics, the following properties are relevant. Whether or not they are fulfilled can make differences in emerging behaviour of the type shown in Fig. 11.3.

Definition 1 (**Properties of combination functions**)

(a) A function c(..) is called *nonnegative* if c(V_1, …, V_k) $\geq$ 0 for all V_1, …, V_k
(b) A function c(..) *respects* 0 if V_1, …, $V_k \geq 0 \Rightarrow$ [c(V_1, …, V_k) = 0 $\Leftrightarrow$ $V_1 = \cdots = V_k = 0$]
(c) A function c(..) is called *monotonically increasing* if

$$U_i \leq V_i \text{ for all } i \Rightarrow \mathrm{c}(U_1, \ldots, U_k) \leq \mathrm{c}(V_1, \ldots, V_k)$$

(d) A function c(..) is called *strictly monotonically increasing* if

$$U_i \leq V_i \text{ for all } i, \text{and } U_j < V_j \text{ for at least one } j \Rightarrow \mathrm{c}(U_1, \ldots, U_k) < \mathrm{c}(V_1, \ldots, V_k)$$

(e) A function c(..) is called *scalar-free* if c(αV_1, …, αV_k) = αc(V_1, …, V_k) for all $\alpha > 0$

In Table 11.3 it is shown which functions have which of the properties (c) to (e) from Definition 1. The following propositions are useful to prove that certain combination functions have the above properties.

Proposition 1 Linear combinations with positive coefficients of functions that are (strictly) *monotonic or scalar-free also are* (strictly) monotonic or scalar-free, respectively.

Proposition 2 Any function composed of monotonically increasing or decreasing functions including an even number of monotonically decreasing functions is monotonically increasing. The same holds for strictly monotonically increasing or decreasing.

Proposition 3 For every $n > 0$ a Euclidean combination function of nth degree is strictly monotonic, scalar-free, symmetric and respects 0.

The properties (a) and (b) are basic properties silently assumed to hold for all combination functions considered here. Sometimes combination functions are defined in such a way that (a) automatically holds:

$$\mathrm{c}^*(V_1, \ldots, V_k) = \begin{array}{ll} \mathrm{c}(V_1, \ldots, V_k) & \text{if } \mathrm{c}(V_1, \ldots, V_k) \geq 0 \\ 0 & \text{otherwise} \end{array}$$

Properties (d) and (e) define a specific class of combination functions; this class includes all Euclidean combination functions and geometric mean combination functions, but logistic sum combination functions do not belong to this class, as they are not scalar-free. Also maximum-based combination functions do not belong to this class as they are monotonic but not strict. A number of results on emerging behaviour will be discussed for this class in particular; note that most functions in this class are nonlinear.

See Table 11.3 for which functions have which of the properties (c) to (e) from Definition 1.

Table 11.3 Characteristics of Definition 1 for the example combination functions

	(c)	(d)	(e)
id(.)	+	+	+
$\mathbf{ssum}_{\lambda}$**(..)**	+	+	+
$\mathbf{eucl}_{n,\lambda}$**(..)**	+	+	+
smin(..)	+	−	+
smax(..)	+	−	+
$\mathbf{sgeomean}_{\lambda}$**(..)** for $V_i > 0$	+	+	+
$\mathbf{alogistic}_{\sigma,\tau}$**(..)**	+	+	−

Proposition 4 (**Proportional outcomes**)
If in a temporal-causal network all combination functions are scalar-free and in some Scenario 1 the initial values for the states are a factor $\boldsymbol{\rho}$ times the initial values in a Scenario 2, then for every t the state values in Scenario 1 are $\boldsymbol{\rho}$ times the corresponding state values in Scenario 2, assuming $\boldsymbol{\eta}_Y \Delta t \leq 1$ for all states Y. This also holds for the equilibrium values when an equilibrium is reached.

Proposition 5 (**Order preservation**)
If in a temporal-causal network all combination functions are monotonically increasing and in some Scenario 1 the initial values for the states are $\leq$ the initial values in a Scenario 2, then for every t the state values in Scenario 1 are $\leq$ the corresponding state values in Scenario 2, assuming $\boldsymbol{\eta}_Y \Delta t \leq 1$ for all states Y. This also holds for the equilibrium values when an equilibrium is reached. Similarly for decreasing.

Definition 2 (**normalised network**)
A network is *normalised* or uses normalised combination functions if for each state Y it holds $\mathrm{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right) = 1$, where $X_1,\ldots, X_k$ are the states from which Y gets its incoming connections.

This normalisation can be achieved in two ways:

(1) **normalisation by adjusting the combination functions**

If any combination function $\mathbf{c}_Y$**(..)** is replaced by $\mathbf{c}'_Y$**(..)** defined as

$$\mathbf{c}'_Y(V_1, \ldots, V_k) = \mathbf{c}_Y(V_1, \ldots, V_k)/\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right) \tag{11.7}$$

then the network is normalised: $\mathbf{c}'_A\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right) = 1$

(2) **normalisation by adjusting the connection weights**

For scalar-free combination functions also normalisation is possible by adapting the connection weights; define

$$\boldsymbol{\omega}'_{X_i,Y} = \boldsymbol{\omega}_{X_i,Y}/\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right) \tag{11.8}$$

Then the network becomes normalised; indeed $\mathbf{c}_Y\left(\boldsymbol{\omega}'_{X_1,Y}, \ldots, \boldsymbol{\omega}'_{X_k,Y}\right) = 1$.

For different example functions, following normalisation (1) above, their normalised variants are given by Table 11.4.

11.4.2 Relevant Network Connectivity Characteristics: Being Strongly Connected

Another important determinant for emerging behaviour is formed by the network's connectivity characteristics, in particular in how far the network has paths connecting any two states:

Definition 3 (**reachable, strongly connected and symmetric network**)

(a) State Y is *reachable* from state X if there is a directed path from X to Y with nonzero connection weights and speed factors.
(b) A network is *connected* if between every two states there is a (nondirected) path with nonzero connection weights and speed factors. It is *strongly connected* if any state Y is reachable from any state X.
(c) A network is *fully connected* if for any states X, Y there is a (direct) connection from X to Y.
(d) A network is called *weakly symmetric* if for all nodes X and Y it holds $\boldsymbol{\omega}_{X,Y} = 0 \Leftrightarrow \boldsymbol{\omega}_{Y,X} = 0$ or, equivalently: $\boldsymbol{\omega}_{X,Y} > 0 \Leftrightarrow \boldsymbol{\omega}_{Y,X} > 0$. The network is called *fully symmetric* if $\boldsymbol{\omega}_{X,Y} = \boldsymbol{\omega}_{Y,X}$ for all nodes X and Y. An adaptive network is called *continually* (*weakly/fully*) *symmetric* if at all time points it is (weakly/fully) symmetric.
(e) A state Y is called *independent* if for any incoming connection with connection weight $\boldsymbol{\omega}_{X,Y} > 0$ the speed factor of Y is 0 (or no incoming connections exist).

Note that an independent state is not reachable from any other state. The term independent means that its behaviour over time is not affected by the other states. Either its value can remain constant (when the speed factor is 0), or it can show any autonomously defined dynamics (see, for example, state X_6 in Fig. 11.4).

Definition 4 (**symmetric combination function**)
A combination function is *symmetric* in a subset S of its arguments if for any $U_1, \ldots, U_k$ is obtained from $V_1, \ldots, V_k$ by a permutation of the arguments in S, it holds $\mathbf{c}(U_1, \ldots, U_k) = \mathbf{c}(V_1, \ldots, V_k)$. It is *fully symmetric* if S is the set of all arguments.

Table 11.4 Normalisation of combination functions

Combination function	Notation	Normalising scaling factor	Normalised combination function
Identity function	**id(.)**	$\boldsymbol{\omega}_{X,Y}$	$V/\boldsymbol{\omega}_{X,Y}$
Scaled sum	$\mathbf{ssum}_{\lambda}(V_1, \ldots, V_k)$	$\boldsymbol{\omega}_{X_1,Y} + \cdots + \boldsymbol{\omega}_{X_k,Y}$	$(V_1 + \cdots + V_k)/(\boldsymbol{\omega}_{X_1,Y} + \cdots + \boldsymbol{\omega}_{X_k,Y})$
Scaled maximum	$\mathbf{smax}_{\lambda}(V_1, \ldots, V_k)$	$\max(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\max(V_1, \ldots, V_k)/\max(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$
Scaled minimum	$\mathbf{smin}_{\lambda}(V_1, \ldots, V_k)$	$\min(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\min(V_1, \ldots, V_k)/\min(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$
Scaled geometric mean	$\mathbf{sgeomean}_{\lambda}(V_1, \ldots, V_k)$	$\boldsymbol{\omega}_{X_1,Y} * \ldots * \boldsymbol{\omega}_{X_k,Y}$	$\sqrt[k]{\frac{V_1 * \ldots * V_k}{\boldsymbol{\omega}_{X_1,Y} * \ldots * \boldsymbol{\omega}_{X_k,Y}}}$
Euclidean	$\mathbf{eucl}_{n,\lambda}(V_1, \ldots, V_k)$	$\boldsymbol{\omega}^n_{X_1,Y} + \cdots + \boldsymbol{\omega}^n_{X_k,Y}$	$\sqrt[n]{\frac{V_1^n + \cdots + V_k^n}{\boldsymbol{\omega}^n_{X_1,Y} + \cdots + \boldsymbol{\omega}^n_{X_k,Y}}}$
Advanced logistic	$\mathbf{alogistic}_{\sigma,\tau}(V_1, \ldots, V_k)$	$\mathbf{alogistic}_{\sigma,\tau}(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\frac{\frac{1}{1+e^{-\sigma(V_1 + \cdots + V_k - \tau)}} - \frac{1}{1+e^{\sigma\tau}}}{\frac{1}{1+e^{-\sigma(\boldsymbol{\omega}_{X_1,Y} + \cdots + \boldsymbol{\omega}_{X_k,Y} - \tau)}} - \frac{1}{1+e^{\sigma\tau}}}$

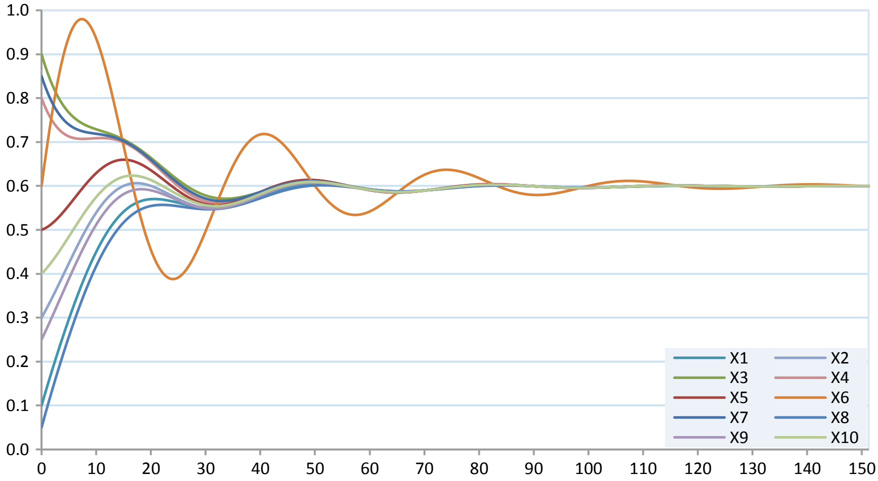

Fig. 11.4 How the dynamics of one independent state X_6 affects all states over time

11.5 Results Relating Emerging Behaviour to Network Structure

This section focuses on the emerging behaviour properties and how they relate to the main structure characteristics connectivity, aggregation, and timing. From these, the first two were discussed in Sect. 11.4, and the third one, timing, will be discussed in Sect. 11.5.4 below. In the current section, it will be shown how properties of these three elements entail properties of emerging behaviour (the horizontal arrow in the upper part of Fig. 11.1). First a few basic definitions and results; see also (Treur 2016a).

11.5.1 Basic Definitions and Results

Definition 5 (stationary point and equilibrium)
A state Y has a *stationary point* at t if $\mathbf{d}Y(t)/\mathbf{d}t = 0$. The network is in *equilibrium* at t if every state Y of the network has a stationary point at t.

Applying this for the specific differential equation format for a temporal-causal network model, a more specific criterion can be formulated in terms of the network structure characteristics $\boldsymbol{\omega}_{X,Y}$, $\mathbf{c}_Y(..)$, $\boldsymbol{\eta}_Y$:

Lemma 1 (Criterion for a stationary point in a temporal-causal network)
Let Y be a state and $X_1, \ldots, X_k$ the states from which state Y gets its incoming connections. Then Y has a stationary point at t if and only if $\boldsymbol{\eta}_Y = 0$ or $\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\right) = Y(t)$.

The following proposition and theorem show that for normalised scalar-free combination functions, always when all states have the same value (for example, initially), an equilibrium occurs. For proofs, see Chap. 15, Sect. 15.6.

Proposition 6 Suppose a network with nonnegative connections has normalised scalar-free combination functions.

(a) If $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections, and $X_1(t) = \cdots = X_k(t) = V$ for some common value V, then also $\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\right) = V$.

(b) If, moreover, the combination functions are monotonic, and $V_1 \leq X_1(t), \ldots, X_k(t) \leq V_2$ for some values V_1 and V_2, then also $V_1 \leq \mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\right) \leq V_2$ and if $\boldsymbol{\eta}_Y\ \Delta t \leq 1$ and $V_1 \leq Y(t) \leq V_2$ then $V_1 \leq Y(t + \Delta t) \leq V_2$.

Theorem 1 (**common state values provide equilibria**)
Suppose a network with nonnegative connections is based on normalised and scalar-free combination functions. Then the following hold.

(a) Whenever all states have the same value V, the network is in an equilibrium state.

(b) If for every state for its initial value V it holds $V_1 \leq V \leq V_2$, then for all t for every state Y it holds $V_1 \leq Y(t) \leq V_2$. In an achieved equilibrium for every state for its equilibrium value V it holds $V_1 \leq V \leq V_2$.

11.5.2 Common Equilibrium Values for Acyclic and Strongly Connected Networks

In this section the focus is on networks with neat connectivity properties, namely being acyclic or being strongly connected. For these types some results are discussed below. However, also for any network connectivity good results are possible, but these results depend on the network's connectivity structure in terms of its strongly connected components. Those more general results are addressed in Chap. 12.

Theorem 1 does not tell whether other types of equilibria, where the values are not the same, are possible as well, for example, as shown in the first and third graph in Fig. 11.3. In subsequent theorems it is shown that in many cases no other types of equilibria occur. As a first case, consider a network without cycles. Then the following theorem has been proven by applying induction over the acyclic graph connections starting from the independent states and thereby using Proposition 6 and Lemma 1.

Theorem 2 (**equilibrium states provide common state values; acyclic case**)
Suppose an acyclic network with nonnegative connections is based on normalised and scalar-free combination functions.

(a) If in an equilibrium state the independent states all have the same value V, then all states have the same value V.
(b) If, moreover, the combination functions are monotonic, and in an equilibrium state the independent states all have values V with $V_1 \leq V \leq V_2$, then all states have values V with $V_1 \leq V \leq V_2$.

Next, a basic Lemma for dynamics of normalised networks with combination functions which are monotonically increasing and scalar-free.

Lemma 2 Let a normalised network with nonnegative connections be given with combination functions that are monotonically increasing and scalar-free; then:

(a)

(i) If for some node Y at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \leq Y(t)$, then $Y(t)$ is decreasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \leq 0$.
(ii) If the combination functions are strictly increasing and a node X exists with $X(t) < Y(t)$ and $\boldsymbol{\omega}_{X,Y} > 0$, and the speed factor of Y is nonzero, then $Y(t)$ is strictly decreasing at t: $\mathbf{d}Y(t)(t)/\mathbf{d}t < 0$.

(b)

(i) If for some node Y at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \geq Y(t)$, then $Y(t)$ is increasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \geq 0$.
(ii) *If*, the combination function is strictly increasing and a node X exists with $X(t) > Y(t)$ and $\boldsymbol{\omega}_{X,Y} > 0$, and the speed factor of Y is nonzero, then $Y(t)$ is strictly increasing at t: $\mathbf{d}Y(t)(t)/\mathbf{d}t > 0$.

Using Lemma 1 and 2 the following proposition has been proven for strongly connected networks with cycles.

Theorem 3 (**common equilibrium state values; strongly connected case**) Suppose the network has normalised, scalar-free and strictly monotonic combination functions, then:

(a) If the network is strongly connected, then in an equilibrium state all states have the same value.
(b) Suppose the network has one or more independent states and the subnetwork without these independent states is strongly connected. If in an equilibrium state all independent states have values V with $V_1 \leq V \leq V_2$, then all states have values V with $V_1 \leq V \leq V_2$. In particular, when all independent states have the same value V, then all states have this same value V.

Using Lemma 1 and 2 the following slightly more general theorem has been proven for (connected) networks with cycles and possibly with an independent state.

Theorem 4 (**equilibrium states provide common state values**) Suppose a (possibly cyclic) network with nonnegative connections is based on normalised, strictly monotonically increasing and scalar-free combination functions, then:

(a) If in an equilibrium state, a state Y with nonzero speed factor has highest state value or lowest state value, then all states X from which Y is reachable have the same equilibrium state value as Y.
(b) Suppose except for at most one independent state, every state Y is reachable from all other states X. Then in an equilibrium state all states have the same state value.
(c) Under the conditions of (b) the equilibrium state is attracting, and the common equilibrium state value is between the highest and lowest previous or initial state values.

Theorems 2, 3 and 4 can be applied to many cases and then prove that all states converge to the same value. For example, this explains why for the second simulation in Fig. 11.3 convergence to one common value takes place, but not for the first and third case. For the first case this is because it does not satisfy the scalar-free condition, and the for the third case because it does not satisfy the condition on reachability: one exceptional independent state is allowed but not two, as occurs in the third example in Fig. 11.3.

As an illustration for another function satisfying the above conditions of being scalar-free and strictly monotonically increasing, in Fig. 11.5 a simulation example is shown for a normalised scaled geometric mean function. This function is indeed scalar free, monotonically increasing, and strictly monotonically increasing as long as the values are nonzero. So by Theorem 3 a common equilibrium value may be expected.

As predicted by Theorems 1 and 3 all state values indeed end up in the same value and this value is between the minimal and maximal initial values.

11.5.3 The Effect of Independent States on a Network's Emerging Behaviour

The one exceptional independent state allowed in Theorem 4(b) can have any independently preset constant value, and all other state values converge to this value. But it is also possible to give this state an autonomous pattern over time that converges to some limit value $\underline{\mathbf{V}}$ for $t \to \infty$. Then over time all state values will more or less follow this pattern and end up in the same equilibrium value $\underline{\mathbf{V}}$, all according to Theorem 4(b); see Fig. 11.4. Therefore:

Corollary 1 Assume the conditions of Theorem 4 hold, and one state X is independent, for which its value over time is described by the function $f(t)$, so $X(t) = f(t)$ for all t. If $\lim_{t\to\infty} f(t) = V$, then this V is the common equilibrium value for all states.

Corollary 1 is illustrated by Fig. 11.5 where X_6 is an independent state and has dynamics based on $f(t) = b_2 + b_1 e^{-a_2 t} \sin(2\pi a_1 t)$ with $a_1 = 0.03$, $a_2 = 0.035$,

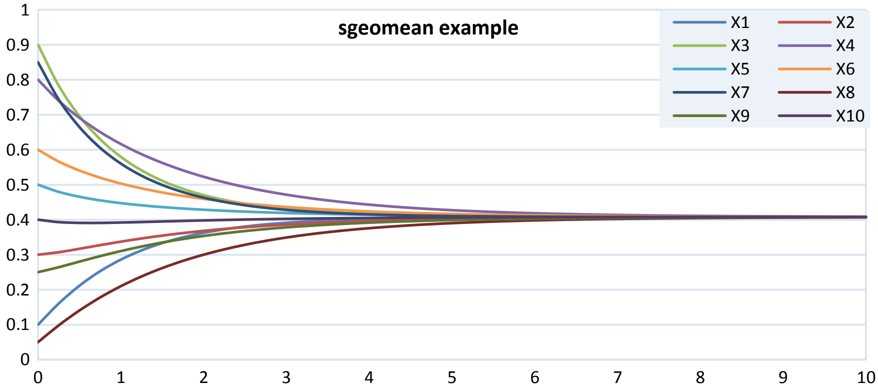

Fig. 11.5 Example simulation for the normalised scaled geometric mean function

$b_1 = 0.5$, $b_2 = 0.6$. Here the outgoing connections of X_6 have been increased according to Table 11.5 to get a stronger effect.

Figure 11.6 shows two cases in which condition (b) of Theorem 4 is not fulfilled: there are two independent states. In the upper graph X_3 and X_7 are constant at values 0.85 and 0.9, respectively, and all other equilibrium values turn out to end up between these values. In the lower graph, X_1 and X_8 are constant at values 0.1 and 0.05, respectively, and also here all other equilibrium values turn out to end up between these values. So, even if these two states both have very low or very high values, still the other state values end up between these values. Note that as in Fig. 11.3c, here the equilibrium values are not equal, although in this case they are close to each other. This is consistent with Theorems 3 and 4.

Theorem 3(b) shows that under the conditions assumed there, all equilibrium value are in between the highest and lowest initial values of independent states, which is illustrated in Fig. 11.6.

More about this can be found when also the role of timing is taken into account as modeled by speed factors.

Table 11.5 Adjusted weights (Compared to Box 11.1) of outgoing connections from X_6

Connection weights	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_6	0.6	0.9	0.7	0.95	0.9		0.8	0.7	0.8	0.9

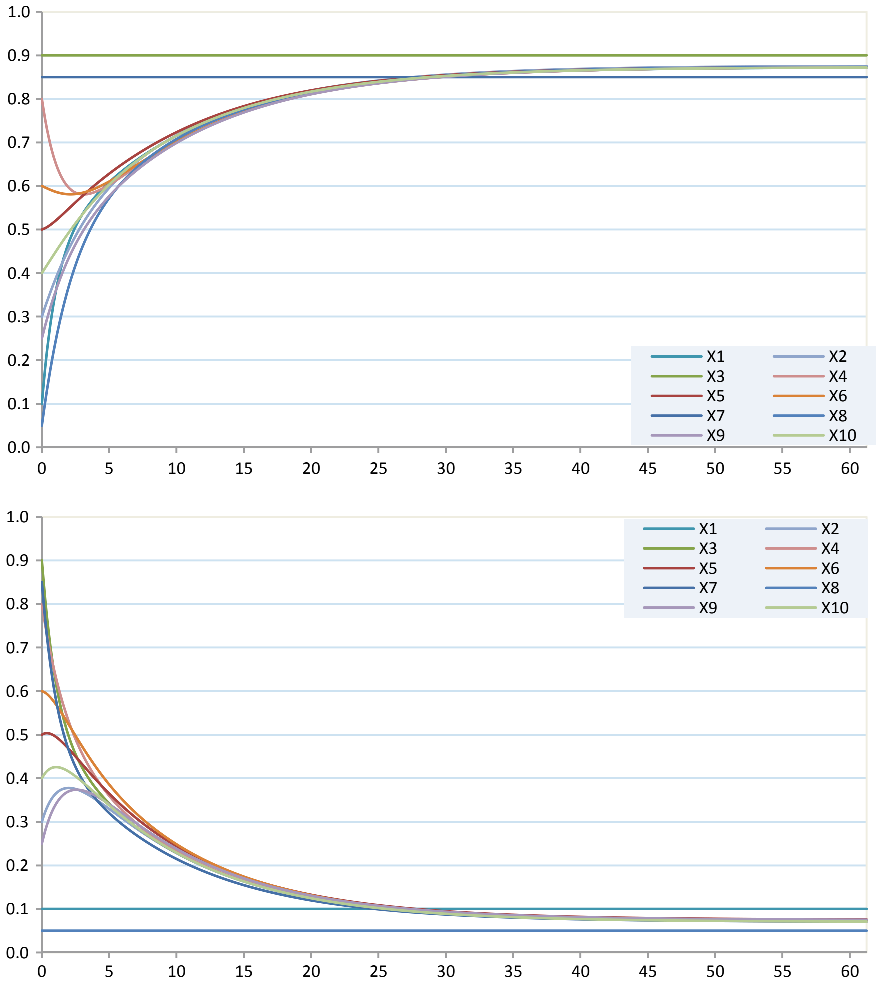

Fig. 11.6 How multiple independent states affect all equilibrium values

11.5.4 How Timing Affects a Common Equilibrium Value

Taking also *timing* into account, as modeled by speed factors, the following Theorem 5 is a further refinement of the above. It shows that under some assumptions any value between the highest and lowest initial value can be the common equilibrium value.

Theorem 5 (**Variability of the common equilibrium value**)
Suppose a connected network with n states and only nonnegative connections is based on normalised, strictly monotonically increasing and scalar-free combination functions, then:

(a) A function $\underline{\mathbf{eqf}}$: $[0, 1]^{2n} \rightarrow [0, 1]$ exists that assigns in an achieved equilibrium the common equilibrium value $\underline{\mathbf{V}}_{X_i} = \underline{\mathbf{V}}$ of the states to the values of the speed factors $\boldsymbol{\eta}_{X_i}$, $i = 1, \ldots, n$, and the initial state values V_{X_i}, $i = 1, \ldots, n$ of all states:

$$\underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right) = \underline{\mathbf{V}}_{X_i} = \underline{\mathbf{V}} \text{ for all } i$$

(b) This function $\underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right)$ is surjective: every value $V \in [0, 1]$ can occur as some achieved equilibrium value $\underline{\mathbf{V}} = \underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right)$. More specifically, it holds:

(i) For any value $V \in [0, 1]$ and any values of speed factors $\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}$, initial values $V_{X_1} \ldots, V_{X_n}$ exist such that

$$\underline{\mathbf{V}} = \underline{\mathbf{eqf}}(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}) = V$$

(ii) For any initial values $V_{X_1} \ldots, V_{X_n}$, values of speed factors $\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}$ exist such that

$$\underline{\mathbf{V}} = \underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right) = V_{X_i}$$

(iii) Moreover, if it is assumed that $\underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right)$ is a continuous function of the speed factors $\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}$, then for any initial values $V_{X_1} \ldots, V_{X_n}$, and any value V with $\min(V_{X_1} \ldots, V_{X_n}) \leq V \leq \max(V_{X_1} \ldots, V_{X_n})$, values of speed factors $\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}$ exist such that

$$\underline{\mathbf{V}} = \underline{\mathbf{eqf}}\left(\boldsymbol{\eta}_{X_1}, \ldots, \boldsymbol{\eta}_{X_n}, V_{X_1} \ldots, V_{X_n}\right) = V$$

Theorem 5 shows that both the initial values and the speed factors affect the common equilibrium value. Note that Theorem 5b) (iii) depends on the assumption that the common equilibrium value is a continuous function of the speed factors. This will depend on characteristics of the combination functions, but it is not clear by which types of combination functions this assumption is satisfied, in addition to being normalised, strictly monotonically increasing and scalar-free combination functions. Should they be continuous? Differentiable? With continuous partial derivatives which are bounded? Or smooth of a certain order, maybe even of infinite order? These are still open questions. However, as illustrated in Fig. 11.7, Theorem 5b) (iii) *at least has been confirmed in simulation experiments*.

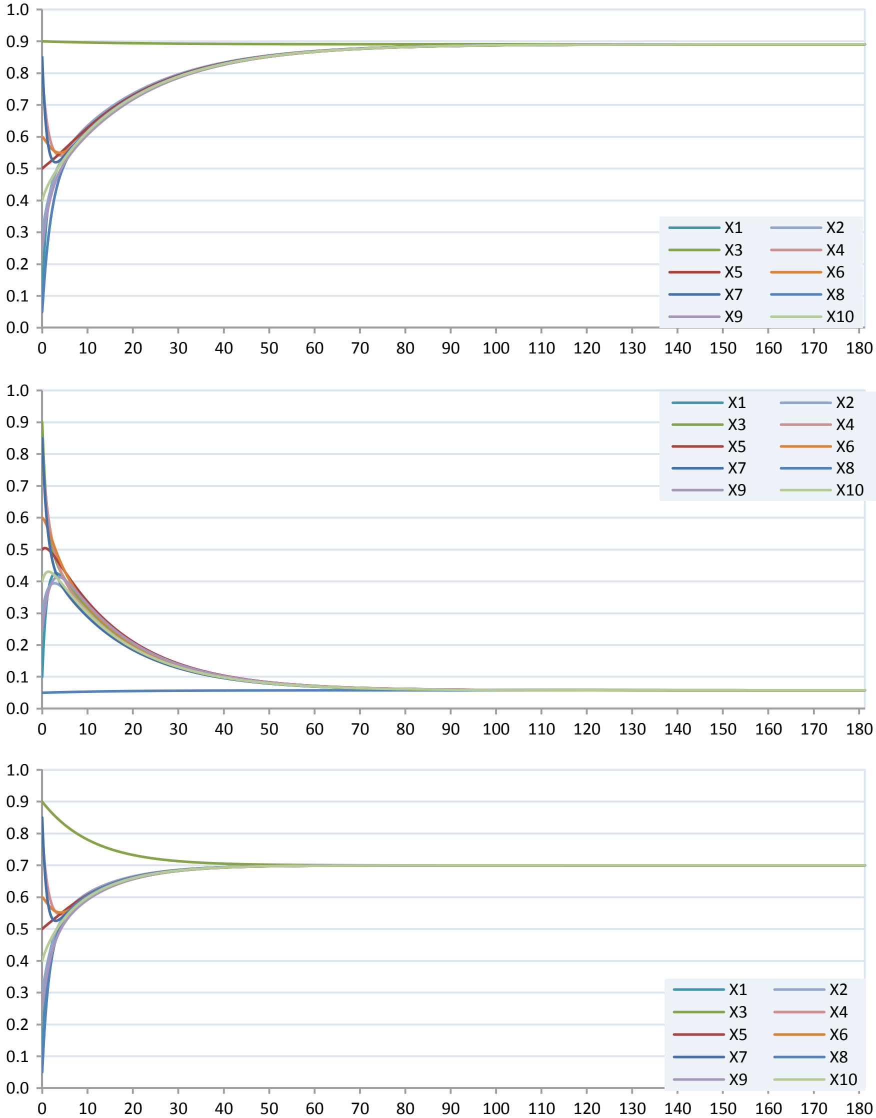

Fig. 11.7 How any value can become a common equilibrium value by using appropriate speed factors (here normalised scaled sum functions were used)

Here for any arbitrary choice for V by adapting values of speed factors in a smooth manner always such values could be determined such that V became the common equilibrium value $\underline{\mathbf{V}}$, thereby clearly showing experimentally that the function **eqf(..)** was continuous.

An example of this for normalised scaled sum functions where two speed factors (of X_3, resp. X_8) *are adapted is shown in* Fig. 11.7. The upper graph shows the pattern when the speed factor of X_3 is 0.001, and the middle graph when the speed factor of X_8 is 0.001. In both cases the common equilibrium value is quite close to the initial value of X_3 *resp.* X_8. Next, an arbitrary value $V = 0.7$ in between was chosen. It was found that for speed factor 0.04 for X_3 and 0.645 for X_8 the common equilibrium value was 0.700001. See lower graph in Fig. 11.7. This illustrates Theorem 5b) (iii).

11.5.5 Emerging Behaviour for Fully Connected Networks

The theorems above all only apply to combination functions that are scalar free, such as Euclidean combination functions and scaled geometric mean combination functions. The advanced logistic sum combination function is also often used; it is not scalar free, so we don't have discussed any results for that type of function yet. But at least some result has been obtained without the scalar free assumption, summarized in Theorem 6. Here the combination functions are assumed symmetric and (not necessarily strictly) monotonically increasing; the logistic sum combination function satisfies that condition. However, in this case there is an additional condition on the connection weights: they all should be equal, and the network should be fully connected. This result still applies to Euclidean combination functions as well, but, also to logistic sum combination functions, and maximum and minimum combination functions, which indeed are symmetric and monotonic combination functions.

Theorem 6 Suppose in a network with nonnegative connections the combination functions are symmetric and monotonically increasing and the network is fully connected with equal weights: $\boldsymbol{\omega}_{X,Y} = \boldsymbol{\omega}$ for all X and Y. Then in an equilibrium state all states have the same value.

11.6 Emerging Behaviour for Euclidean, Scaled Maximum and Scaled Minimum Combination Functions

In Sect. 11.3 example simulations were discussed only for scaled sum, advanced logistic sum and scaled geometric mean combination functions. In the current section also scaled maximum and minimum and Euclidean combination functions of different orders are discussed in some more depth. At the end a relationship between normalised Euclidean combination functions and normalised maximum combination functions is found in simulations and mathematically proven. The example used is the same as in Sect. 11.3 with characteristics and initial values shown in Box 11.1 and Table 11.2. Step size was $\Delta t = 0.25$.

11.6.1 Emerging Behaviour for Euclidean Combination Functions of Different Orders

In this section simulations for the Euclidean combination functions of varying order n are discussed, for higher values $n = 2$, 4, 10 and 100 (see Figs. 11.8 and 11.9), and for a lower value $n = 0.0001$ (see Fig. 11.12). The following theorem is for now only a conjecture, as a proof of it is complicated that increasing n leads to increasing outcomes.

Theorem 7 (strictly monotonically increasing trend of eucl(n, V_1, …, V_k) for n) For equal state values between 0 and 1, the equilibrium values for normalised Euclidean combination functions have a strictly monotonically increasing trend as function of the order n.

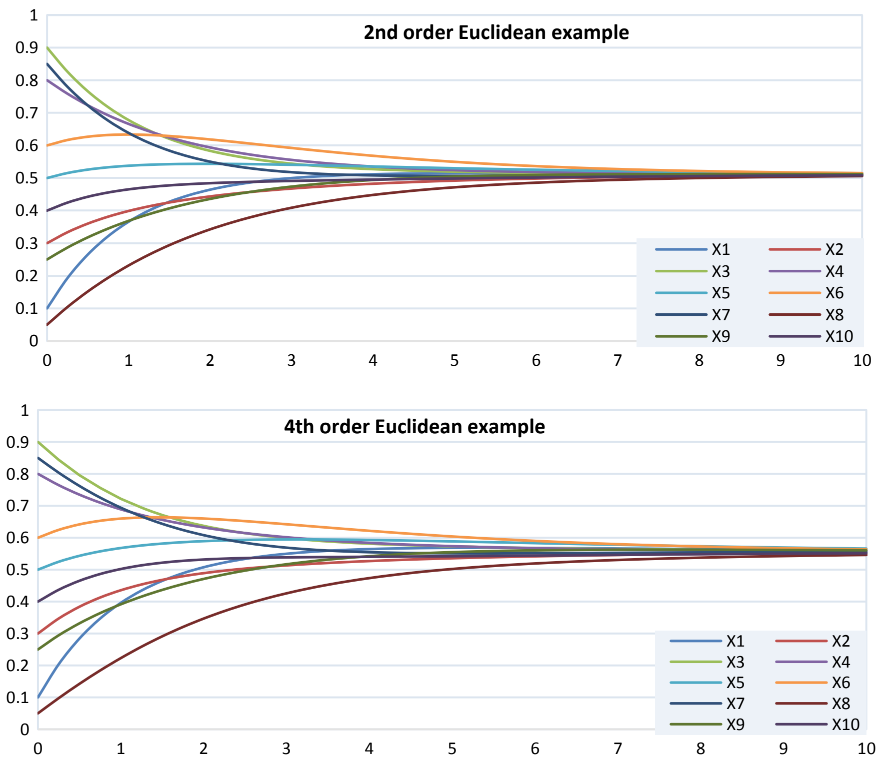

Fig. 11.8 Example simulations for the normalised Euclidean functions of order 2 and 4

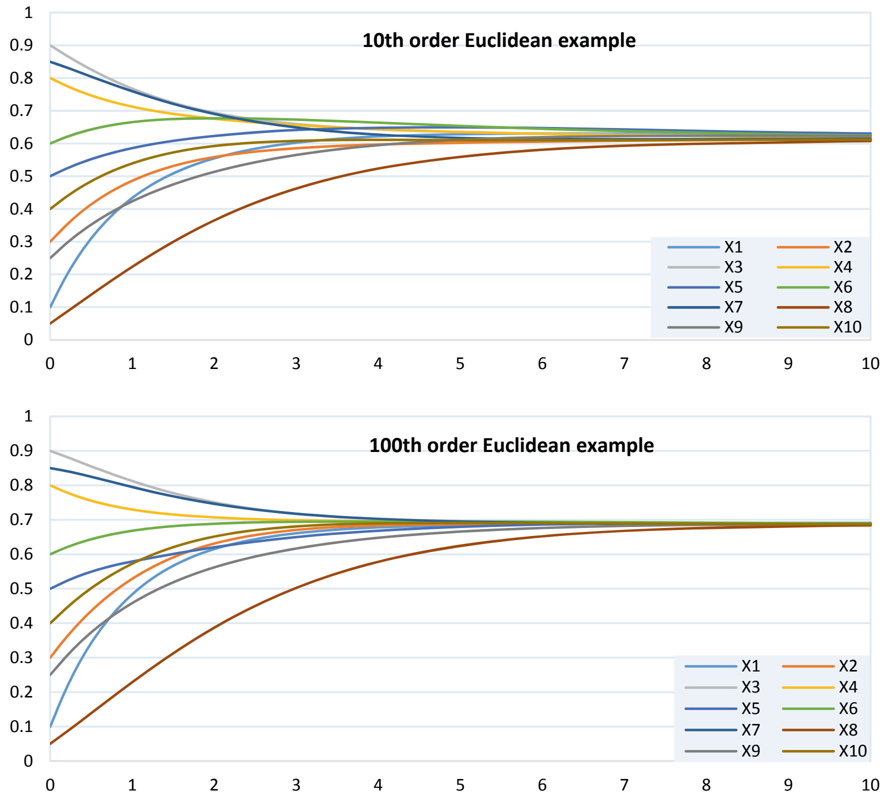

Fig. 11.9 Example simulations for the normalised Euclidean functions of order 10 and 100

Theorem 7 is confirmed by the simulation examples shown in Figs. 11.8, 11.9 and 11.12.

In Fig. 11.8 the behavior is as expected from Theorems 1 and 3. Note that the fourth order function gets a higher common final value than the second order, while the second order one gets almost the same value as the first order one in Fig. 11.3. To explore what happens if the order is increased further, also the 10th order and 100th order Euclidean functions were simulated; see Fig. 11.9.

The final value indeed increases with the order n. Will it still increase further until 1? An answer for this comes in Sect. 11.6.2 when it is compared with the simulation for the normalised scaled maximum function; see Fig. 11.10.

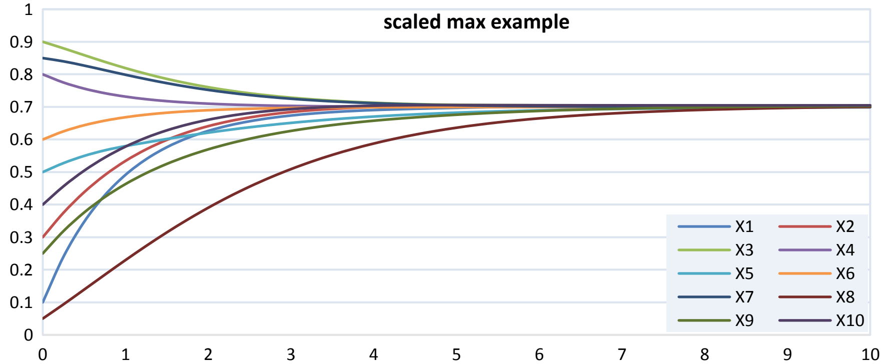

Fig. 11.10 Example simulation for the normalised scaled maximum function

11.6.2 Comparing Equilibrium Values for Euclidean Combination Functions and Scaled Maximum Combination Functions

Note that this pattern shown in Fig. 11.10 is not predicted by Theorem 3 or 4 as the scaled maximum function is not strictly monotonically increasing. But it also does not contradict these theorems as they do not formulate an if and only if relation.

Maybe a bit surprisingly, the pattern of the normalised maximum combination function is very similar to the pattern for the 100th order Euclidean combination function. Indeed, it has been found by mathematical proof that when the order n is increased, the nth order Euclidean function approximates the normalised scaled max function; see Theorem 8 below. To prove this, first in Lemma 3 a general mathematical relation between radical and max expressions is shown.

Lemma 3 (**Relating radical and max expressions**)
Suppose $a_1, .., a_k$ are any nonnegative real numbers. Then

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \cdots + a_k^n} = \max(a_1, \ldots, a_k)$$

Theorem 8 Let for each n the normalised Euclidean combination function $\mathbf{eucl}_{n,\lambda(n)}(V_1, \ldots, V_k)$ be given with normalising factor $\lambda(n)$, and let the normalised scaled maximum combination function $\mathbf{smax}_\lambda(V_1, \ldots, V_k)$ be given with scaling factor λ. Then for all $V_1, \ldots, V_k$ it holds

$$\lim_{n\to\infty} \mathbf{eucl}_{n,\lambda(n)}(V_1, \ldots, V_k) = \mathbf{smax}_\lambda(V_1, \ldots, V_k)$$

where

$$\lambda(n) = \boldsymbol{\omega}^n_{X_1,Y} + \cdots + \boldsymbol{\omega}^n_{X_k,Y}$$

and

$$\lambda = \max\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right)$$

For proofs, see Chap. 15, Sect. 15.6.

This Theorem 8 gives some kind of hint for why the graphs for the normalised 100th order Euclidean combination function and for the normalised scaled maximum combination function in Figs. 11.9 and 11.10 are very similar: they are almost the same combination function. It also suggests that for higher orders than 100 the final value will not become much higher, although Theorem 7 is no exact proof for this point.

Returning to the issue that the scaled maximum function does not fulfil the requirement of being strictly monotonically increasing, indeed there are cases in which the conclusions of Theorem 3 do not apply. An example is the slightly adjusted network shown in Box 11.2. Here this time the mutual connections between X_1, X_2, and X_3, and the mutual connections between X_8, X_9, and X_{10} all have weight 1. The other weights remain the same.

Box 11.2 Role matrices showing adjusted mutual connection weights for X_1, X_2, and X_3, and for X_8, X_9, and X_{10}

mb base connectivity	1	2	3	4	5	6	7	8	9
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_1	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_3	X_1	X_2	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_4	X_1	X_2	X_3	X_5	X_6	X_7	X_8	X_9	X_{10}
X_5	X_1	X_2	X_3	X_4	X_6	X_7	X_8	X_9	X_{10}
X_6	X_1	X_2	X_3	X_4	X_5	X_7	X_8	X_9	X_{10}
X_7	X_1	X_2	X_3	X_4	X_5	X_6	X_8	X_9	X_{10}
X_8	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_9	X_{10}
X_9	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_{10}
X_{10}	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9

mcw connection weights	1	2	3	4	5	6	7	8	9
X_1	1	1	0.25	0.25	0.25	0.2	0.1	0.25	0.2
X_2	1	1	0.15	0.2	0.1	0.1	0.25	0.15	0.25
X_3	1	1	0.25	0.1	0.25	0.2	0.1	0.25	0.2
X_4	0.1	0.2	0.1	0.2	0.25	0.15	0.25	0.15	0.2
X_5	0.2	0.1	0.2	0.15	0.25	0.2	0.05	0.2	0.1
X_6	0.15	0.2	0.15	0.8	0.25	0.2	0.15	0.1	0.2
X_7	0.1	0.15	0.1	0.25	0.2	0.1	0.25	0.2	0.15
X_8	0.25	0.25	0.25	0.15	0.1	0.25	0.2	1	1
X_9	0.25	0.25	0.1	0.25	0.2	0.25	0.15	1	1
X_{10}	0.1	0.25	0.15	0.25	0.15	0.1	0.25	1	1

In Fig. 11.11 simulation results are shown for the normalised scaled maximum combination function. Now clustering takes place with the first group X_1, X_2, and X_3 ending up in value 0.5415, the second group X_8, X_9, and X_{10} in value 0.325, and the remaining group X_4, X_5, X_6 and X_7 in exactly 0.7. This shows that indeed

Theorem 3 is not applicable for scaled maximum combination functions (because that function is not strictly monotonous).

Note that for practical reasons of limited machine precision it is less straightforward to simulate this modified network for a 100th order Euclidean combination function. The reason is that the 100th powers of the single impacts $V_{i,j} = \boldsymbol{\omega}_{Xi,Xj}\, X_j(t)$ from different states X_i on X_j are of very different order of magnitude. They may easily be a factor up to 10^{60} or more different, for example, and when added in ${V_{1,j}}^{100} + \cdots + {V_{k,j}}^{100}$ to form the aggregated impact

$$\sqrt[100]{\frac{{V_{1,j}}^{100} + \cdots + {V_{k,j}}^{100}}{\boldsymbol{\lambda}}}$$

the contributions of the smaller terms are vanishing due to limited machine precision. So, then it is as if for X_1, X_2, and X_3, and for X_8, X_9, and X_{10} a network is simulated with only the connection weights 1 and the rest 0, so then clusters show up in an artificial manner. To simulate this in a correct manner, machine precision of at least 60 digits would be needed.

Returning to the Euclidean combination function and the original network connectivity depicted in Box 11.1, also very small positive real numbers for n can be explored. Figure 11.12 shows a simulation graph for $n = 0.0001$. It turns out that this graph is practically equal to the one for the normalised scaled minimum combination function. There may be a theorem similar to Theorem 8 but then for $n \to 0$ and the minimum operator to explain this. For now, this is left open.

All in all, it turns out that by varying the order n from (close to) 0 to ∞, the common equilibrium value for the states of the example network varies with n from around 0.4 to around 0.7, where these boundaries are the values reached for the normalised scaled minimum and maximum combination functions.

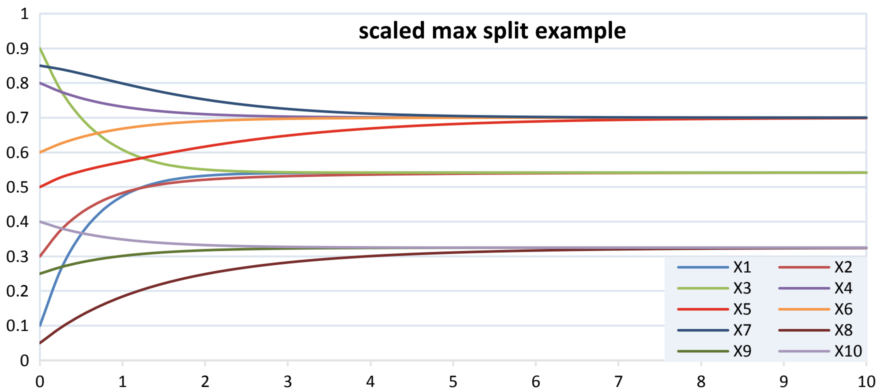

Fig. 11.11 Example simulation for the normalised scaled maximum function for the modified network of Box 11.2

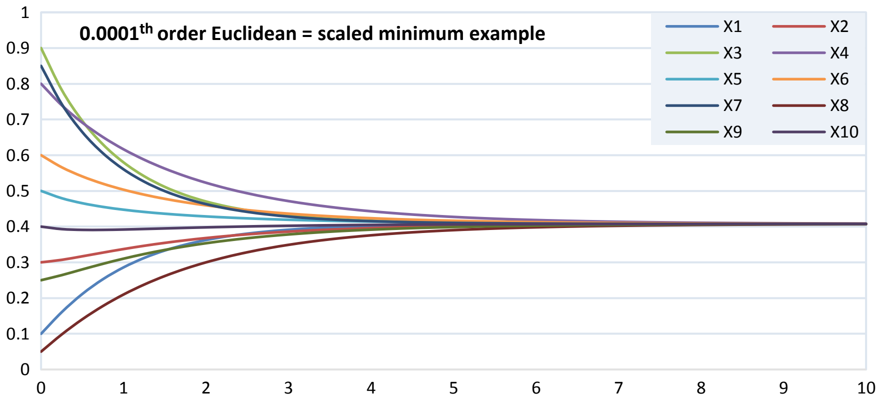

Fig. 11.12 Example simulation for the normalised Euclidean functions of order 0.0001, which turns out to be practically equal to the outcome for the normalised scaled minimum function

11.7 Discussion

Emerging behaviour of a network is a consequence of the characteristics of the network structure, although it can be challenging to find out how this relation between structure and behaviour exactly is. In this chapter a number of theorems and proofs were presented on how certain identified properties of network structure determine network behaviour, in particular concerning equilibrium values reached. Parts of this chapter were adopted from (Treur 2018a). The considered network structure characteristics include the main elements network *connectivity characteristics* (in how far other states of the network are reachable from a given state), network *aggregation characteristics* in terms of properties of combination functions used to aggregate multiple impacts on a single state, and network *timing characteristics* in terms of speed factors for the states. This challenge was addressed using a Network-Oriented Modeling approach based on temporal-causal network models (Treur 2016b, 2019), and the analysis they allow, as a vehicle; e.g., (Treur 2017). Within Mathematics the analysis of emerging behaviour of dynamical systems has a long history that goes back to (Picard 1891; Poincaré 1881); see also (Brauer and Nohel 1969; Lotka 1956).

In the network literature the challenge to relate network behaviour to network structure is usually only addressed for specific models and functions, where often these functions are assumed linear. In the current chapter it was addressed in a more general way for general properties of network structure covering a variety of models or functions, also including various nonlinear functions such as *n*th order Euclidean combination functions and scaled geometric mean combination functions. In this way extra insight was obtained in what properties exactly make that specific network structures lead to certain network behaviour.

As an example, a special case of Theorem 3, for one specific model with one specific combination function and speed factor, and just for fully connected networks appeared in (Bosse et al. 2015). In that paper it does not become clear what it exactly is that makes that the considered model generates that type of emerging behaviour. The much more general formulation presented and proven in Theorem 3 here is new. It shows that the structure-behaviour relation depends on the one hand on *connectivity* of the network, but on the other hand also on *aggregation* (the ways in which multiple impacts on a single state are aggregated), and on *timing* (how fast states respond on impact they receive). It has been found that aggregation structure properties for the combination functions such as strict monotonicity and being scalar-free are crucial. These properties define a relatively wide class of functions including linear (scaled sum) functions but mostly also nonlinear functions such as Euclidean functions and product-based (scaled) geometric mean functions. It also has been shown that networks with examples of combination functions not in this class, such as logistic sum combination functions (not scalar-free) and maximum-based combination functions (not strictly monotonic) do not show the same behaviour.

Theorems 4 and 5 are also new and explore the bounds and range for the equilibrium values reached. Theorem 6 first appeared in (Treur 2017) in the context of an adaptive network to model a specific preferential attachment principle, but actually covers the much more general setting as provided in the current chapter. In (Hendrickx and Tsitsiklis 2013) also analysis of emerging behaviour was addressed, but for a different class of networks.

Concerning connectivity, the work presented in the current chapter mainly addresses strongly connected networks. As a next step, more general networks have been addressed that may or may not be strongly connected. Analysis has been performed based on such a network's strongly connected components. Results have been found for any network, without any condition on strong connectivity; see Chap. 12 or (Treur 2018b).

In other further work, also for adaptive networks a similar challenge has been addressed: how do certain properties of adaptation principles lead to certain types of adaptive network behaviour. For example, adaptation based on a Hebbian learning principle [see Chap. 14 or (Treur 2018d)], or based on a homophily principle [see Chap. 13 or (Treur 2018c)] have been analysed, and properties of the combination functions used have been identified that lead to certain expected behaviour for the adaptive connection weights.

References

Bosse, T., Duell, R., Memon, Z.A., Treur, J., van der Wal, C.N.: Agent-based modeling of emotion contagion in groups. Cogn. Comput. **7**, 111–136 (2015)

Brauer, F., Nohel, J.A.: Qualitative Theory of Ordinary Differential Equations. Benjamin (1969)

Castellano, C., Fortunato, S., Loreto, V.: Statistical physics of social dynamics. Rev. Mod. Phys. **81**, 591–646 (2009)

Hendrickx, J.M., Tsitsiklis, J.N.: Convergence of type-symmetric and cut-balanced consensus seeking systems. IEEE Trans. Autom. Control **58**(1), 214–218. (2013) Extended version: arXiv:1102.2361v2[cs.SY]

Lotka, A.J.: Elements of Physical Biology. Williams and Wilkins Co. (1924), 2nd ed. Dover Publications (1956)

Pearl, J.: Causality. Cambridge University Press, Cambridge (2000)

Picard, E.: Traité d'Analyse, vol. 1 (1891), vol. 2 (1893)

Poincaré, H.: Mémoire sur les courbes défine par une équation différentielle (On curves defined by differential equations) (1881–1882) (1881)

Treur, J.: Verification of temporal-causal network models by mathematical analysis. Vietnam J. Comput. Sci. **3**, 207–221 (2016a)

Treur, J.: Network-Oriented Modeling: Addressing Complexity of Cognitive, Affective and Social Interactions. Springer Publishers (2016b)

Treur, J.: Modelling and analysis of the dynamics of adaptive temporal-causal network models for evolving social interactions. Comput. Soc. Netw. **4**, article 4, 1–20 (2017)

Treur, J.: Relating emerging network behaviour to network structure. In: Proceedings of the 7th International Conference on Complex Networks and their Applications, Complex Networks' 18, vol. 1. Studies in Computational Intelligence, vol. 812, pp. 619–634. Springer Publishers (2018a)

Treur, J.: Mathematical analysis of a network's asymptotic behaviour based on its strongly connected components. In: Proceedings of the 7th International Conference on Complex Networks and their Applications, Complex Networks' 18, vol. 1. Studies in Computational Intelligence, vol. 812, pp. 663–679. Springer Publishers (2018b)

Treur, J.: Relating an adaptive social network's structure to its emerging behaviour based on homophily. In: Proceedings of the 7th International Conference on Complex Networks and their Applications, Complex Networks' 18. Springer, vol. 2. Studies in Computational Intelligence, vol. 813, pp. 341–356. Springer Publishers (2018c)

Treur, J.: Relating an adaptive network's structure to its emerging behaviour for Hebbian learning. In: Martín-Vide, C., Vega-Rodríguez, M.A., Fagan, D., O'Neill, M. (eds.), Theory and Practice of Natural Computing: 7th International Conference, TPNC 2018. Lecture Notes in Computer Science, vol. 11324, pp. 359–373. Springer Publishers (2018d)

Treur, J.: The ins and outs of network-oriented modeling: from biological networks and mental networks to social networks and beyond. Trans. Comput. Collective Intell. **32**, 120–139. Keynote Lecture at ICCCI'18 (2019)